# A New Approach to Signed Eulerian Numbers

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**Abstract**

The numbers of even and odd permutations with a given ascent number are investigated by an operator that was introduced in [9]. Their difference is called a signed Eulerian number. By means of the operator the recurrence relation for signed Eulerian numbers can be deduced, which was obtained in [1] by an analytic method. Our approach is straightforward and enables us to deduce other properties including divisibility properties by prime powers.
AMS Subject Classification: 05A05, 20B30.

## 1. Introduction

An *ascent* (or *descent*) of a permutation $a_1a_2\cdots a_n$ of $[n] = \{1, 2, \ldots, n\}$ is an adjacent pair such that $a_i < a_{i+1}$ (or $a_i > a_{i+1}$) for some $i$ $(1 \le i \le n-1)$. Let $E(n,k)$ be the set of all permutations of $[n]$ with exactly $k$ ascents, where $0 \le k \le n-1$. Its cardinality is the classical *Eulerian number*;

$$A_{n,k} = |E(n,k)|,$$

whose properties and identities can be found in [2-6], for example.

An *inversion* of a permutation $A = a_1a_2\cdots a_n$ is a pair $(i,j)$ such that $1 \le i < j \le n$ and $a_i > a_j$. Let us denote by $\mathrm{inv}(A)$ the number of inversions in a permutation $A$, and by $E_{\mathrm{e}}(n,k)$ or $E_{\mathrm{o}}(n,k)$ the subsets of all permutations in $E(n,k)$ that have, respectively, even or odd numbers of inversions. The aim of this paper is to investigate their cardinalities;

$$B_{n,k} = |E_{\mathrm{e}}(n,k)| \quad \text{and} \quad C_{n,k} = |E_{\mathrm{o}}(n,k)|.$$

Obviously we have $A_{n,k} = B_{n,k} + C_{n,k}$, while the differences

$$D_{n,k} = B_{n,k} - C_{n,k}$$

were called *signed Eulerian numbers* in [1], where descents of permutations were considered instead of ascents. Therefore, the identities for $D_{n,k}$ presented here correspond to those in [1] that are obtained by replacing $k$ with $n-1-k$.

In order to study these numbers, we make use of an operator on permutations in $[n]$, which was introduced in [9]. In the subsequent papers [10] and [11], it was shown that the operator plays a relevant role in studying Eulerian numbers. The operator $\sigma$ is defined by adding one to all entries of a permutation and by changing $n+1$ into one. However, when $n$ appears at either end of a permutation, it is removed and one is put at the other end. That is, for a permutation $a_1a_2\cdots a_n$ with $a_i = n$ for some $i$ $(2 \le i \le n-1)$, we have

(i) $\sigma(a_1a_2\cdots a_n) = b_1b_2\cdots b_n$,

where $b_i = a_i + 1$ for all $i$ $(1 \le i \le n)$ and $n+1$ is replaced by one. And, for a permutation $a_1a_2\cdots a_{n-1}$ of $[n-1]$, we have:

(ii) $\sigma(a_1a_2\cdots a_{n-1}n) = 1b_1b_2\cdots b_{n-1}$;

(iii) $\sigma(na_1a_2\cdots a_{n-1}) = b_1b_2\cdots b_{n-1}1$,

where $b_i = a_i + 1$ for all $i$ $(1 \le i \le n-1)$. We denote by $\sigma^\ell A$ the repeated $\ell$ applications of $\sigma$ to a permutation $A$.

It is obvious that the operator preserves the number of ascents or descents in a permutation, that is, $\sigma A \in E(n,k)$ if and only if $A \in E(n,k)$. Let us observe the number of inversions of a permutation when $\sigma$ is applied.

When $n$ appears at either end of a permutation $A = a_1a_2\cdots a_n$ as in (ii) or (iii), it is evident that

$$\mathrm{inv}(\sigma A) = \mathrm{inv}(A).$$

Now let us consider the case (i). When $a_i = n$ for some $i$ $(2 \le i \le n-1)$, we get $\sigma(a_1a_2\cdots a_n) = b_1b_2\cdots b_n$, where $b_i = 1$ is at the position. In this case, $n-i$ inversions $(i, i+1), \ldots, (i, n)$ of $A$ vanish and, in turn, $i-1$ inversions $(1, i), \ldots, (i-1, i)$ of $\sigma A$ occur. Hence the difference between the numbers of inversions is

$$\mathrm{inv}(\sigma A) - \mathrm{inv}(A) = (i-1) - (n-i) = 2i - (n+1). \tag{1}$$

Therefore, when $n$ is even, each application of the operator changes the parity of permutations as long as $n$ remains in the interior of permutations. If $n$ is odd, however, the operator $\sigma$ also preserves the parity of all permutations of $[n]$.

For convenience sake we denote by $E_\mathrm{e}^-(n,k)$ and $E_\mathrm{e}^+(n,k)$ the sets of permutations $a_1a_2\cdots a_n$ in $E_\mathrm{e}(n,k)$ with $a_1 < a_n$ and $a_1 > a_n$, respectively. Similarly, $E_\mathrm{o}^-(n,k)$ and $E_\mathrm{o}^+(n,k)$ denote those in $E_\mathrm{o}(n,k)$. In $E_\mathrm{e}^-(n,k)$ or $E_\mathrm{o}^-(n,k)$ *canonical* permutations are those of the form $1a_2a_3\cdots a_n$, and in $E_\mathrm{e}^+(n,k)$ or $E_\mathrm{o}^+(n,k)$ are those of the form $a_2a_3\cdots a_n1$, where $a_2a_3\cdots a_n$ is a permutation of $\{2, 3, \ldots, n\}$.

In [8] and references therein, even or odd permutations were classified by anti-excedance number, not by the ascent number. An anti-excedance in a permutation $A = a_1a_2\cdots a_n$ means an inequality $i \ge a_i$. Recurrence relations were also given for the cardinalities of the sets of even and odd permutations that are classified by the anti-excedance number. The recurrence relations held for all $n$. The classification of even or odd permutations by the ascent number seems not so simple, as will be seen in the following sections.

The signed Eulerian numbers $D_{n,k} = B_{n,k} - C_{n,k}$, however, have a recurrence relation that holds for all $n$, although it has different expressions according to the parity of $n$. The relation was conjectured in [7] and an analytic proof for it was given in [1]. In Section 4 we will derive it from a quite different point of view based on the properties of the operator $\sigma$.

## 2. The Numbers $B_{n,k}$ and $C_{n,k}$

The numbers $B_{n,k}$ and $C_{n,k}$ enjoy some symmetry properties according to the values of $n$. The permutation $n\cdots 21 \in E(n,0)$ has $n(n-1)/2$ inversions. Hence the values of $B_{n,0}$ and $C_{n,0}$ are given by

$$B_{n,0} = \begin{cases} 1, & \text{if } n \equiv 0 \text{ or } 1 \pmod 4, \\ 0, & \text{if } n \equiv 2 \text{ or } 3 \pmod 4, \end{cases}$$

and

$$C_{n,0} = \begin{cases} 0, & \text{if } n \equiv 0 \text{ or } 1 \pmod 4, \\ 1, & \text{if } n \equiv 2 \text{ or } 3 \pmod 4. \end{cases}$$

For a permutation $A = a_1a_2\cdots a_n$ we define its reflection by $A^* = a_n\cdots a_2a_1$. Using reflected permutations and the parity of $n(n-1)/2$, the following symmetry between $B_{n,k}$ and $C_{n,k}$ are easily checked.

(i) $n \equiv 0$ or $1 \pmod 4$. In this case, $A \in E_{\mathrm{e}}(n,k)$ if and only if $A^* \in E_{\mathrm{e}}(n, n-k-1)$, and $A \in E_{\mathrm{o}}(n,k)$ if and only if $A^* \in E_{\mathrm{o}}(n, n-k-1)$, so we have

$$B_{n,k} = B_{n,n-k-1} \text{ and } C_{n,k} = C_{n,n-k-1}.$$

(ii) $n \equiv 2$ or $3 \pmod 4$. In this case, $A \in E_{\mathrm{e}}(n,k)$ if and only if $A^* \in E_{\mathrm{o}}(n, n-k-1)$, and $A \in E_{\mathrm{o}}(n,k)$ if and only if $A^* \in E_{\mathrm{e}}(n, n-k-1)$, so we have

$$B_{n,k} = C_{n,n-k-1} \text{ and } C_{n,k} = B_{n,n-k-1}.$$

The values of $B_{n,k}$ and $C_{n,k}$ for small $n$ are shown in the next two tables. The integers in their top rows represent the values of $k$. In Section 4 a formula for calculating these numbers will be supplied by means of $A_{n,k}$ and $D_{n,k}$.

| $B_{n,k}$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| $n=2$ | 0 | 1 | | | | | | | | |
| $n=3$ | 0 | 2 | 1 | | | | | | | |
| $n=4$ | 1 | 5 | 5 | 1 | | | | | | |
| $n=5$ | 1 | 14 | 30 | 14 | 1 | | | | | |
| $n=6$ | 0 | 28 | 155 | 147 | 29 | 1 | | | | |
| $n=7$ | 0 | 56 | 605 | 1208 | 586 | 64 | 1 | | | |
| $n=8$ | 1 | 127 | 2133 | 7819 | 7819 | 2133 | 127 | 1 | | |
| $n=9$ | 1 | 262 | 7288 | 44074 | 78190 | 44074 | 7288 | 262 | 1 | |
| $n=10$ | 0 | 496 | 23947 | 227623 | 655039 | 655315 | 227569 | 23893 | 517 | 1 |

| $C_{n,k}$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| $n=2$ | 1 | 0 | | | | | | | | |
| $n=3$ | 1 | 2 | 0 | | | | | | | |
| $n=4$ | 0 | 6 | 6 | 0 | | | | | | |
| $n=5$ | 0 | 12 | 36 | 12 | 0 | | | | | |
| $n=6$ | 1 | 29 | 147 | 155 | 28 | 0 | | | | |
| $n=7$ | 1 | 64 | 586 | 1208 | 605 | 56 | 0 | | | |
| $n=8$ | 0 | 120 | 2160 | 7800 | 7800 | 2160 | 120 | 0 | | |
| $n=9$ | 0 | 240 | 7320 | 44160 | 78000 | 44160 | 7320 | 240 | 0 | |
| $n=10$ | 1 | 517 | 23893 | 227569 | 655315 | 655039 | 227623 | 23947 | 496 | 0 |

## 3. The Case of Odd $n$

Throughout this section we assume that $n$ is an odd integer. In this case, the orbit of a permutation of $E_{\mathrm{e}}^-(n,k)$ under $\sigma$ is entirely contained in $E_{\mathrm{e}}^-(n,k)$ and similarly for $E_{\mathrm{e}}^+(n,k)$, as was shown in Section 1. Here we mainly deal only with the set $E_{\mathrm{e}}(n,k)$ and its cardinality $B_{n,k}$, for the same arguments can also be applied to $E_{\mathrm{o}}(n,k)$ and its cardinality $C_{n,k}$.

It was shown in [9] that to each permutation $A$ there corresponds a smallest positive integer $\pi(A)$ such that $\sigma^{\pi(A)}A = A$, which is called the *period* of $A$. Its trace

$$\{\sigma A, \sigma^2 A, \ldots, \sigma^{\pi(A)}A = A\}$$

is called the *orbit* of $A$. Also there it was shown that the period satisfies the relation

$$\pi(A) = \begin{cases} (n-k)\gcd(n, \pi(A)) & \text{if } A \in E^-(n,k), \\ (k+1)\gcd(n, \pi(A)) & \text{if } A \in E^+(n,k). \end{cases} \tag{2}$$

It follows from (2) that the period of a permutation $A \in E(n,k)$ is either $d(n-k)$ or $d(k+1)$ for a positive divisor $d$ of $n$, i.e., $d = \gcd(n, \pi(A))$, although there may be no permutations having such periods for some divisors. In this paper, divisors of $n$ always mean positive divisors.

For a divisor $d$ of $n$, we denote by $\alpha_d^k$ the number of orbits of period $d(n-k)$ in $E_{\mathrm{e}}^-(n,k)$ and by $\beta_d^k$ that of orbits of period $d(k+1)$ in $E_{\mathrm{e}}^+(n,k)$. In the case of odd $n$ the next theorem plays a fundamental role.

**Theorem 3.1.** Let $n$ be an odd integer and let $k$ be an integer satisfying $1 \le k \le n-1$. Then it follows that

$$B_{n-1,k-1} = \sum_{d|n} d\alpha_d^k, \tag{3}$$

$$B_{n-1,k} = \sum_{d|n} d\beta_d^k, \tag{4}$$

$$B_{n,k} = \sum_{d|n} d\{(n-k)\alpha_d^k + (k+1)\beta_d^k\}. \tag{5}$$

*Proof.* First let us consider permutations in $E_{\mathrm{e}}^-(n,k)$. Since each orbit contains at least one canonical permutation, it suffices to deal only with canonical ones in counting orbits. If $A = 1a_2a_3\cdots a_n \in E_{\mathrm{e}}^-(n,k)$, we see that $(a_2-1)(a_3-1)\cdots(a_n-1) \in E_{\mathrm{e}}(n-1,k-1)$, since

$$\mathrm{inv}((a_2-1)(a_3-1)\cdots(a_n-1)) = \mathrm{inv}(A)$$

and one is deleted. Therefore, there are $B_{n-1,k-1}$ canonical permutations in $E_{\mathrm{e}}^-(n,k)$. It follows from (2) that the period of a permutation $A \in E_{\mathrm{e}}^-(n,k)$ is equal to $d(n-k)$ for a divisor $d$ of $n$. There exist $n$ canonical permutations in $\{\sigma A, \sigma^2 A, \ldots, \sigma^{n(n-k)}A = A\}$ due to [9, Corollary 2], and hence each orbit $\{\sigma A, \sigma^2 A, \ldots, \sigma^{d(n-k)}A = A\}$ of a permutation $A$ with period $d(n-k)$ contains exactly $d$ canonical permutations. This follows from the fact that the latter repeates itself $n/d$ times in the former. Since there exist $\alpha_d^k$ orbits of period $d(n-k)$ for each divisor $d$ of $n$, classifying all canonical permutations of $E_{\mathrm{e}}^-(n,k)$ into orbits leads us to (3).

The proof of (4) is similar. To do this we consider permutations in $E_{\mathrm{e}}^+(n,k)$. If $A = a_2a_3\cdots a_n1 \in E_{\mathrm{e}}^+(n,k)$, we see that $(a_2-1)(a_3-1)\cdots(a_n-1) \in E_{\mathrm{e}}(n-1,k)$, since

$$\mathrm{inv}((a_2-1)(a_3-1)\cdots(a_n-1)) = \mathrm{inv}(A) - (n-1)$$

and $n-1$ is an even number by assumption. Therefore, the set of all canonical permutations in $E_{\mathrm{e}}^+(n,k)$ has cardinality $B_{n-1,k}$. Again using (2), the period of a permutation $A \in E_{\mathrm{e}}^+(n,k)$ is equal to $d(k+1)$ for a divisor $d$ of $n$. By [9, Corollary 2] there exist $n$ canonical permutations in $\{\sigma A, \sigma^2 A, \ldots, \sigma^{n(k+1)}A = A\}$ and hence, as above, there exist exactly $d$ such permutations in each orbit $\{\sigma A, \sigma^2 A, \ldots, \sigma^{d(k+1)}A = A\}$ of a permutation $A$ with period $d(k+1)$. There exist $\beta_d^k$ orbits of period $d(k+1)$ for each divisor $d$ of $n$. Hence, we can obtain (4) by classifying all canonical permutations in $E_{\mathrm{e}}^+(n,k)$ into orbits.

Considering the numbers of orbits and periods, we see that the cardinalities of $E_{\mathrm{e}}^{\pm}(n,k)$ are obtained by

$$|E_{\mathrm{e}}^-(n,k)| = \sum_{d|n} d(n-k)\alpha_d^k \quad \text{and} \quad |E_{\mathrm{e}}^+(n,k)| = \sum_{d|n} d(k+1)\beta_d^k. \tag{6}$$

Since the set $E_{\mathrm{e}}(n,k)$ is a disjoint union of $E_{\mathrm{e}}^{-}(n,k)$ and $E_{\mathrm{e}}^{+}(n,k)$, we conclude that

$$B_{n,k} = |E_{\mathrm{e}}^{-}(n,k)| + |E_{\mathrm{e}}^{+}(n,k)| = \sum_{d|n} d(n-k)\alpha_d^k + \sum_{d|n} d(k+1)\beta_d^k,$$

which proves (5).

Let us denote by $\gamma_d^k$ the number of orbits of period $d(n-k)$ in $E_{\mathrm{o}}^{-}(n,k)$ and by $\delta_d^k$ that of orbits of period $d(k+1)$ in $E_{\mathrm{o}}^{+}(n,k)$. When $n$ is odd, analogous relations to (3)-(6) hold for $C_{n,k}$, $\gamma_d^k$ and $\delta_d^k$, since the orbit of a permutation of $E_{\mathrm{o}}^{\pm}(n,k)$ under $\sigma$ is also contained in $E_{\mathrm{o}}^{\pm}(n,k)$. We state them for the sake of completeness:

$$\begin{aligned} C_{n-1,k-1} &= \sum_{d|n} d\gamma_d^k; \\ C_{n-1,k} &= \sum_{d|n} d\delta_d^k; \\ C_{n,k} &= \sum_{d|n} d\{(n-k)\gamma_d^k + (k+1)\delta_d^k\}; \end{aligned}$$

and

$$|E_{\mathrm{o}}^{-}(n,k)| = \sum_{d|n} d(n-k)\gamma_d^k \quad \text{and} \quad |E_{\mathrm{o}}^{+}(n,k)| = \sum_{d|n} d(k+1)\delta_d^k.$$

Making use of (3) and (4), we see that both cardinalities in (6) can be written simply by $B_{n,k}$ and their counterparts for $E_{\mathrm{o}}(n,k)$ also follow from the above relations in a similar manner.

**Corollary 3.2.** When $n$ is odd, the cardinalities of $E_{\mathrm{e}}^{\pm}(n,k)$ and $E_{\mathrm{o}}^{\pm}(n,k)$ are given by

(i) $|E_{\mathrm{e}}^{-}(n,k)| = (n-k)B_{n-1,k-1}$ and $|E_{\mathrm{o}}^{-}(n,k)| = (n-k)C_{n-1,k-1}$ $(1 \le k \le n-1)$,

(ii) $|E_{\mathrm{e}}^{+}(n,k)| = (k+1)B_{n-1,k}$ and $|E_{\mathrm{o}}^{+}(n,k)| = (k+1)C_{n-1,k}$ $(0 \le k \le n-2)$.

From these equalities we can obtain the following two corollaries. The relations in Corollary 3.3 have the same form as the recurrence relation for classical Eulerian numbers $A_{n,k}$;

$$A_{n,k} = (n-k)A_{n-1,k-1} + (k+1)A_{n-1,k}. \tag{7}$$

The formula for $C_{n,k}$ can also be obtained from that for $B_{n,k}$ using the equality $A_{n,k} = B_{n,k} + C_{n,k}$.

**Corollary 3.3.** When $n$ is odd, the following relations hold for $B_{n,k}$ and $C_{n,k}$:

$$B_{n,k} = (n-k)B_{n-1,k-1} + (k+1)B_{n-1,k}; \tag{8}$$
$$C_{n,k} = (n-k)C_{n-1,k-1} + (k+1)C_{n-1,k}. \tag{9}$$

**Corollary 3.4.** When $n$ is odd, the following relations hold:

(i) $|E_{\mathrm{e}}^{-}(n,k)| - |E_{\mathrm{o}}^{-}(n,k)| = (n-k)D_{n-1,k-1}$ $(1 \le k \le n-1)$;

(ii) $|E_{\mathrm{e}}^{+}(n,k)| - |E_{\mathrm{o}}^{+}(n,k)| = (k+1)D_{n-1,k}$ $(0 \le k \le n-2)$.

## 4. Recurrence Relation for Signed Eulerian Numbers $D_{n,k}$

When $n$ is even, equality (8) nor (9) does not hold, as is seen from the tables of Section 2. For example, an odd integer $C_{10,4}$ cannot be written as a linear sum of $C_{9,k}$'s or $B_{9,k}$'s $(1 \le k \le 7)$ with integral coefficients, since they are all even. Therefore, in reality (8) nor (9) does not provide a recurrence relation of the numbers $B_{n,k}$ or $C_{n,k}$.

As for the differences $D_{n,k} = B_{n,k} - C_{n,k}$, however, their recurrence relation was conjectured in [7] and an analytic proof for it was given in [1]. In our notation it is described as the next theorem, for which we provide another proof from a combinatorial point of view. Notice that there is a different flavor in the case of even $n$.

**Theorem 4.1.** The recurrence relation for $D_{n,k}$ is given by

$$D_{n,k} = \begin{cases} (n-k)D_{n-1,k-1} + (k+1)D_{n-1,k} & \text{if } n \text{ is odd,} \\ D_{n-1,k-1} - D_{n-1,k} & \text{if } n \text{ is even.} \end{cases} \tag{10}$$

*Proof.* The first part of this relation follows immediately from (8) and (9) of Corollary 3.3. Assuming that $n$ is even, we show the second part by means of the operator $\sigma$.

Recall that when $n$ is even, the operator $\sigma$ may change the parity of permutations of $E(n,k)$ and it is a bijection on $E_{\mathrm{e}}^{-}(n,k) \cup E_{\mathrm{o}}^{-}(n,k)$ and on $E_{\mathrm{e}}^{+}(n,k) \cup E_{\mathrm{o}}^{+}(n,k)$.

First let us consider permutations $A = a_1 a_2 \cdots a_n$ in $E_{\mathrm{e}}^{-}(n,k) \cup E_{\mathrm{o}}^{-}(n,k)$ and divide all permutations in $E_{\mathrm{e}}^{-}(n,k) \cup E_{\mathrm{o}}^{-}(n,k)$ into the following two types:

(i) $A = a_1 a_2 \cdots a_{n-1} n$, where $a_1 a_2 \cdots a_{n-1}$ is a permutation of $[n-1]$;

(ii) $A = a_1 a_2 \cdots a_n$ with $a_1 < a_n$, where $a_i = n$ for some $i$ $(2 \le i \le n-1)$.

Suppose $A \in E_{\mathrm{e}}^{-}(n,k)$. If $A$ is of type (i), then $\sigma A$ remains an even permutation, since $\mathrm{inv}(\sigma A) = \mathrm{inv}(A)$. We see that the cardinality of permutations of type (i) is $B_{n-1,k-1}$, since $A$ is even and $n$ is the last entry. However, if $A \in E_{\mathrm{e}}^{-}(n,k)$ is of type (ii), then we have $\sigma A \in E_{\mathrm{o}}^{-}(n,k)$ by (1). Therefore, the cardinality of permutations of type (ii) in $E_{\mathrm{e}}^{-}(n,k)$ is

$$|E_{\mathrm{e}}^{-}(n,k)| - B_{n-1,k-1},$$

and precisely so many permutations change the parity from even to odd under $\sigma$.

Simliarly, suppose $A \in E_{\mathrm{o}}^{-}(n,k)$. If $A$ is of type (i), then $\sigma A$ remains an odd permutation. We see that the cardinality of permutations of type (i) is $C_{n-1,k-1}$. If $A \in E_{\mathrm{o}}^{-}(n,k)$ is of type (ii) by (1), then we have $\sigma A \in E_{\mathrm{e}}^{-}(n,k)$. The cardinality of permutations of type (ii) in $E_{\mathrm{o}}^{-}(n,k)$ is

$$|E_{\mathrm{o}}^{-}(n,k)| - C_{n-1,k-1},$$

and precisely so many permutations change the parity from odd to even under $\sigma$.

Since $\sigma$ is a bijection on $E_{\mathrm{e}}^{-}(n,k) \cup E_{\mathrm{o}}^{-}(n,k)$, both cardinalities must be equal. Hence we obtain

$$|E_{\mathrm{e}}^{-}(n,k)| - |E_{\mathrm{o}}^{-}(n,k)| = B_{n-1,k-1} - C_{n-1,k-1} = D_{n-1,k-1}. \tag{11}$$

Next let us consider permutations $A = a_1 a_2 \cdots a_n$ in $E_{\mathrm{e}}^{+}(n,k) \cup E_{\mathrm{o}}^{+}(n,k)$ and divide all permutations in $E_{\mathrm{e}}^{+}(n,k) \cup E_{\mathrm{o}}^{+}(n,k)$ into the following two types:

(iii) $A = n a_1 a_2 \cdots a_{n-1}$, where $a_1 a_2 \cdots a_{n-1}$ is a permutation of $[n-1]$;

(iv) $A = a_1 a_2 \cdots a_n$ with $a_1 > a_n$, where $a_i = n$ for some $i$ $(2 \le i \le n-1)$.

If $A \in E_{\mathrm{e}}^{+}(n,k)$ is of type (iii), then $\sigma A$ remains an even permutation. We see that the cardinality of permutations of type (iii) is $C_{n-1,k}$, since $\mathrm{inv}(A) - \mathrm{inv}(a_1 a_2 \cdots a_{n-1}) = n-1$ and $n-1$ is odd. However, if $A \in E_{\mathrm{e}}^{+}(n,k)$ is of type (iv), then we have $\sigma A \in E_{\mathrm{o}}^{+}(n,k)$ by (1). The cardinality of permutations of type (iv) in $E_{\mathrm{e}}^{+}(n,k)$ is

$$|E_{\mathrm{e}}^{+}(n,k)| - C_{n-1,k},$$

and precisely so many permutations change the parity from even to odd under $\sigma$.

Similarly, if $A \in E_{\mathrm{o}}^{+}(n,k)$ is of type (iii), then $\sigma A$ remains an odd permutation. We see that the cardinality of permutations of type (iii) is $B_{n-1,k}$ as above. If $A \in E_{\mathrm{o}}^{+}(n,k)$ is of type (iv), then we have $\sigma A \in E_{\mathrm{e}}^{+}(n,k)$. The cardinality of permutations of type (iv) in $E_{\mathrm{o}}^{+}(n,k)$ is

$$|E_{\mathrm{o}}^{+}(n,k)| - B_{n-1,k},$$

and precisely so many permutations change the parity from odd to even under $\sigma$.

Since $\sigma$ is a bijection on $E_{\mathrm{e}}^{+}(n,k) \cup E_{\mathrm{o}}^{+}(n,k)$, both cardinalities must be equal. Hence we obtain

$$|E_{\mathrm{e}}^{+}(n,k)| - |E_{\mathrm{o}}^{+}(n,k)| = -B_{n-1,k} + C_{n-1,k} = -D_{n-1,k}. \tag{12}$$

Adding (11) and (12) yields

$$B_{n,k} - C_{n,k} = D_{n,k} = D_{n-1,k-1} - D_{n-1,k},$$

which is the required relation. This completes the proof.

Symmetry properties for them follow from the relations presented in Section 2:

(i) For $n \equiv 0$ or $1 \pmod 4$, $D_{n,k} = D_{n,n-k-1}$;

(ii) For $n \equiv 2$ or $3 \pmod 4$, $D_{n,k} = -D_{n,n-k-1}$.

A table for the values of $D_{n,k}$ is given below.

| $D_{n,k}$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|
| $n=2$ | $-1$ | 1 | | | | | | | | |
| $n=3$ | $-1$ | 0 | 1 | | | | | | | |
| $n=4$ | 1 | $-1$ | $-1$ | 1 | | | | | | |
| $n=5$ | 1 | 2 | $-6$ | 2 | 1 | | | | | |
| $n=6$ | $-1$ | $-1$ | 8 | $-8$ | 1 | 1 | | | | |
| $n=7$ | $-1$ | $-8$ | 19 | 0 | $-19$ | 8 | 1 | | | |
| $n=8$ | 1 | 7 | $-27$ | 19 | 19 | $-27$ | 7 | 1 | | |
| $n=9$ | 1 | 22 | $-32$ | $-86$ | 190 | $-86$ | $-32$ | 22 | 1 | |
| $n=10$ | $-1$ | $-21$ | 54 | 54 | $-276$ | 276 | $-54$ | $-54$ | 21 | 1 |

Thus the values of $B_{n,k}$ and $C_{n,k}$ can be known through

$$B_{n,k} = \frac{A_{n,k} + D_{n,k}}{2}, \quad C_{n,k} = \frac{A_{n,k} - D_{n,k}}{2},$$

using $A_{n,k}$ and $D_{n,k}$ that are calculated according to the respective recurrence relations (7) and (10). From these equalities, we can obtain the expressions of $B_{n,k}$ and $C_{n,k}$ by means of $B_{n-1,k}$'s and $C_{n-1,k}$'s in the case of even $n$, which is a counterpart of Corollary 3.3.

**Corollary 4.2.** When $n$ is even, the following relations hold for $B_{n,k}$ and $C_{n,k}$:

$$\begin{aligned}2B_{n,k} &= (n-k+1)B_{n-1,k-1} + kB_{n-1,k} + (n-k-1)C_{n-1,k-1} + (k+2)C_{n-1,k};\\ 2C_{n,k} &= (n-k+1)C_{n-1,k-1} + kC_{n-1,k} + (n-k-1)B_{n-1,k-1} + (k+2)B_{n-1,k}.\end{aligned}$$

From (11) and (12) we get a counterpart of Corollary 3.4.

**Corollary 4.3.** When $n$ is even, the following relations hold:

(i) $|E_{\mathrm{e}}^{-}(n,k)| - |E_{\mathrm{o}}^{-}(n,k)| = D_{n-1,k-1} \quad (1 \le k \le n-1)$;

(ii) $|E_{\mathrm{e}}^{+}(n,k)| - |E_{\mathrm{o}}^{+}(n,k)| = -D_{n-1,k} \quad (0 \le k \le n-2)$.

## 5. Numbers of Orbits and Applications

Again $n$ is assumed to be an odd integer. In this section we derive the numbers of orbits of particular types, and moreover deduce divisibility properties for $B_{n,k}$, $C_{n,k}$, $D_{n,k}$ and related numbers by prime powers from them.

For a positive integer $\ell$ with $\gcd(\ell, n) = 1$, a canonical permutation of $[n]$ of the form

$$P_n^{\ell} = 1(1+\ell)(1+2\ell)\cdots(1+(n-1)\ell)$$

can be defined, where $\ell, 2\ell, \ldots, (n-1)\ell$ represent numbers modulo $n$. According to whether $P_n^{\ell}$ is an even or odd permutation, let us put

$$\epsilon_n^{\ell} = \begin{cases} 1 & \text{if } P_n^{\ell} \text{ is even,} \\ 0 & \text{if } P_n^{\ell} \text{ is odd.} \end{cases}$$

**Theorem 5.1.** Let $n$ be an odd integer and let $k$ be an integer such that $1 \le k \le n-1$.

(i) If a divisor $d$ of $n$ satisfies $\gcd(k, n/d) > 1$, then $\alpha_d^k = \gamma_d^k = 0$.

(ii) If $\gcd(k, n) = 1$, then $\alpha_1^k = \epsilon_n^{n-k}$ and $\gamma_1^k = 1 - \epsilon_n^{n-k}$.

*Proof.* In order to prove (i), suppose $A$ is a permutation that belongs to $E_{\mathrm{e}}^{-}(n,k)$. From (2) its period $\pi(A)$ satisfies $\pi(A) = (n-k)\gcd(n, \pi(A))$. Then, putting $d = \gcd(n, \pi(A))$, we have $\pi(A) = d(n-k)$ and $d = \gcd(n, d(n-k))$, which implies $\gcd(n-k, n/d) = 1$ or $\gcd(k, n/d) = 1$. Consequently, we see that there exist no permutations of period $d(n-k)$, i.e., $\alpha_d^k = 0$, if a divisor $d$ of $n$ satisfies $\gcd(k, n/d) > 1$. The same arguments can be applied to permutations in $E_{\mathrm{o}}^{-}(n,k)$ and we obtain the assertion that $\gamma_d^k = 0$ if $d$ satisfies $\gcd(k, n/d) > 1$.

Next suppose $\gcd(k, n) = 1$. By [9, Theorem 7] we see that there exists a unique orbit of period $n-k$ in $E_{\mathrm{e}}^{-}(n,k) \cup E_{\mathrm{o}}^{-}(n,k)$, which contains only one canonical permutation $P_n^{n-k}$. Hence, if it is an even permutation, then we have $\alpha_1^k = 1$ and $\gamma_1^k = 0$. Otherwise, $\alpha_1^k = 0$ and $\gamma_1^k = 1$. This completes the proof.

From Theorem 5.1 we can derive a criterion under which $B_{n-1,k-1}$ and $C_{n-1,k-1}$ is divisible by a prime power.

**Corollary 5.2.** Let $p$ be a prime and let an odd integer $n$ be divisible by $p^m$ for a positive integer $m$. If $k$ is divisible by $p$, then $B_{n-1,k-1}$, $C_{n-1,k-1}$ and $D_{n-1,k-1}$ are also divisible by $p^m$.

*Proof.* Without loss of generality we can assume that $m$ is the largest integer for which $p^m$ divides $n$. Suppose $k$ is a multiple of $p$. In Theorem 5.1 we have seen that $\alpha_d^k = 0$ for a divisor $d$ of $n$ such that $\gcd(k, n/d) > 1$. On the other hand, it follows that a divisor $d$ for which $\gcd(k, n/d) = 1$ must be a multiple of $p^m$, since $k$ is a multiple of $p$. Therefore, equality (3) of Theorem 3.1 implies that $B_{n-1,k-1}$ is divisible by $p^m$. Similarly, $C_{n-1,k-1}$ is also divisible by $p^m$, if $k$ is a multiple of $p$.

The final corollary easily follows from Corollaries 3.2 and 5.2.

**Corollary 5.3.** Under the same assumptions as Corollary 5.2 it follows that:

(i) If $k$ is divisible by $p^i$ for some $i$ ($1 \leq i \leq m$), then $|E_\mathrm{e}^-(n,k)|$ and $|E_\mathrm{o}^-(n,k)|$ are divisible by $p^{m+i}$.

(ii) If $k+1$ is divisible by $p^i$ for some $i$ ($i \geq 1$), then $|E_\mathrm{e}^+(n,k)|$ and $|E_\mathrm{o}^+(n,k)|$ are divisible by $p^{m+i}$.

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